ECE 447: Robotics Engineering

Lecture 4: Rigid Motions and Homogeneous Transformations (Part 2)

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Lecture Outline:

1. Rotational Transformations.
2. Composition of Rotations.
3. Homogeneous Transformation.
4. Parameterization of Rotations.

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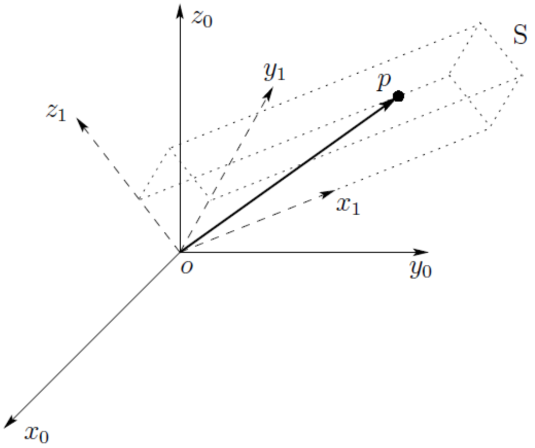
Rotational Transformations:

- The $\{0\}$-frame is our fixed frame, the $\{1\}$-frame is fixed to a rigid body.
- What will happen with points of body (let say p) if we rotate the body, i.e. the $\{1\}$-frame?
- The coordinates of point p in the 1-frame are constant p^1, but in the 0-frame they are changed.
- The coordinates of the point p in 0-frame is:

$$p^0 = R_1^0 \, p^1$$



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Composition of Rotations:

[1] **Rotation about Current Frame:**
Suppose that we have 3 frames:

$$\{0\} = (o_0, x_0, y_0, z_0)$$

$$\{1\} = (o_1, x_1, y_1, z_1)$$

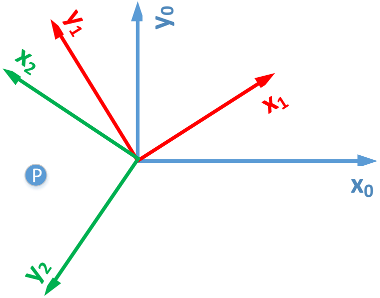
$$\{2\} = (o_2, x_2, y_2, z_2)$$

Any point p can be represented in any of the three coordinates:

$$p^0 = R_1^0 \, p^1$$

$$p^1 = R_2^1 \, p^2$$

$$p^0 = R_2^0 \, p^2$$



Composition of Rotations:

[1] Rotation about Current Frame:

$$p^0 = R_1^0 \; p^1$$

$$p^1 = R_2^1 \; p^2$$

$$p^0 = R_2^0 \; p^2$$

We can write:

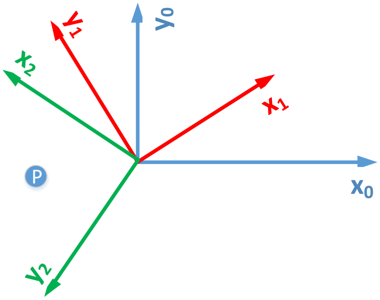
$$p^0 = R_1^0 \; p^1 = R_1^0 \; R_2^1 \; p^2$$

$$p^0 = R_1^0 \; R_2^1 \; p^2$$

$$p^0 = R_2^0 \; p^2$$

$$\boxed{R_2^0 = R_1^0 \; R_2^1}$$

Law of composite rotation

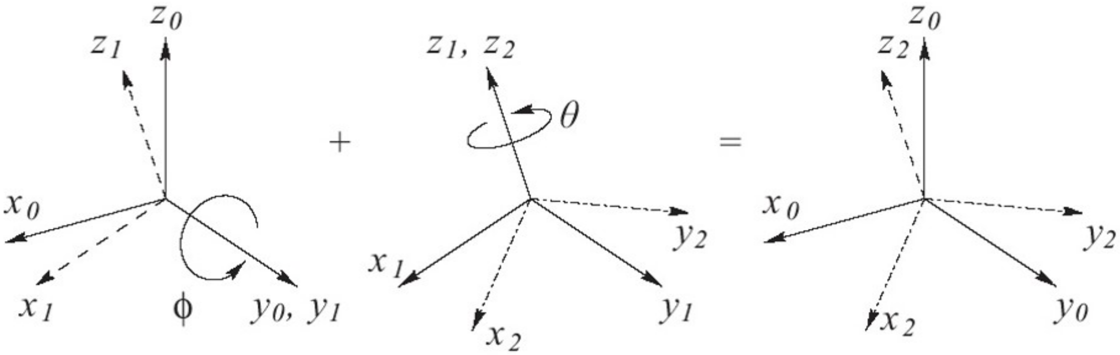


Composition of Rotations:

[1] **Rotation about Current Frame:**

Example: Suppose we rotate:

1. first the frame by angle ϕ around **current** y-axis,
2. then rotate by angle θ around the **current** z-axis. Find the combined rotation ?



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Example: Suppose we rotate:

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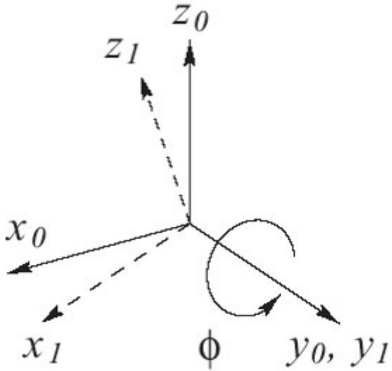
$$R_{y,\phi} = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$



Composition of Rotations:

[1] **Rotation about Current Frame:**

Example: Suppose we rotate:

1. first the frame by angle ϕ around **current** y-axis,
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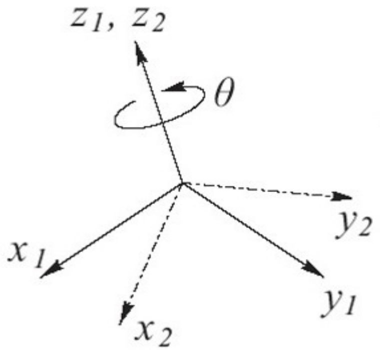
$$R_{z,\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Composition of Rotations:

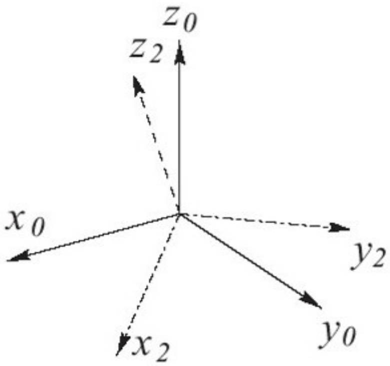
[1] **Rotation about Current Frame:**

$$R_2^0 = R_{y,\phi}\ R_{z,\theta}$$

$$R_2^0 = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2^0 = \begin{bmatrix} c_\phi c_\theta & -c_\phi s_\theta & s_\phi \\ s_\theta & c_\theta & 0 \\ -s_\phi c_\theta & s_\phi c_\theta & c_\phi \end{bmatrix}$$

Note: $s_\phi = \sin(\phi)$ and $c_\theta = \cos(\theta)$



Composition of Rotations:

[1] **Rotation about Current Frame:**

Important Observation: Rotations do not commute.

$$R_{y,\phi}\ R_{z,\theta} \neq R_{z,\theta}\ R_{y,\phi}$$

So that the **order of rotations** is important!

Composition of Rotations:

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Important Observation: Rotations do not commute.

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So that the **order of rotations** is important!

Rule of composite rotation around the current (new) frame:

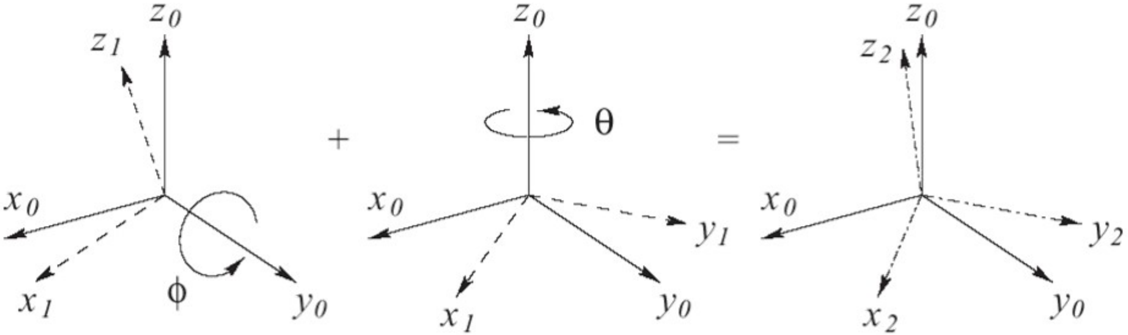
For successive rotations about the current reference frame we use the **post-multiplication** to find the total rotation matrix.

Composition of Rotations:

[2] **Rotation with respect to Fixed Frame:**

Example: Suppose we rotate:

1. the first rotation is by angle ϕ around y_0-axis.
2. then, a rotation by angle θ around z_0-axis (**not z_1-axis**). What is the total rotation ?



Composition of Rotations:

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Example: Suppose we rotate:

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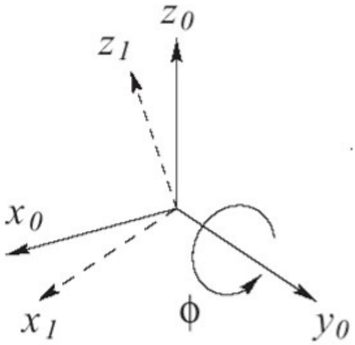
$$R_{y_0,\phi} = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$



Composition of Rotations:

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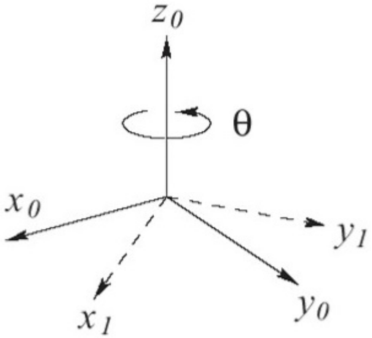
$$R_{z_0,\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

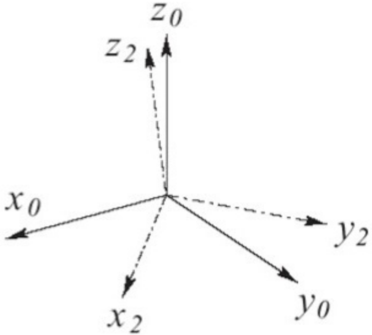


Composition of Rotations:

[2] **Rotation with respect to Fixed Frame:**

$$R_2^0 = R_{z,\theta}\ R_{y,\phi}$$ note the order!

$$R_2^0 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$



Composition of Rotations:

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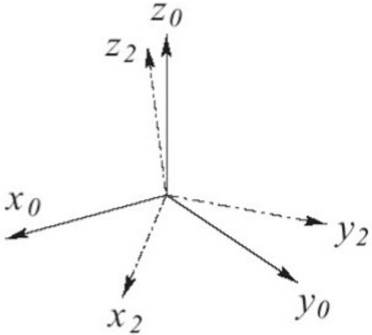
$$R_2^0 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$



Rule of composite rotation around the fixed (original) frame:

For successive rotations about the fixed reference frame we use the **pre-multiplication** to find the total rotation matrix.

Composition of Rotations:

Around fixed frame ? **Pre-multiply**	**Around current frame ?** **Post-multiply**

Example:

Find the rotation R defined by the following basic rotations:

1. A rotation of θ about the current axis x;
2. A rotation of ϕ about the current axis z;
3. A rotation of α about the fixed axis z;
4. A rotation of β about the current axis y;
5. A rotation of δ about the fixed axis x.

Composition of Rotations:

Around fixed frame ?
Pre-multiply

Around current frame ?
Post-multiply

Example:

Find the rotation R defined by the following basic rotations:

1. A rotation of θ about the current axis x;
2. A rotation of ϕ about the current axis z;
3. A rotation of α about the fixed axis z;
4. A rotation of β about the current axis y;
5. A rotation of δ about the fixed axis x.

Solution:

$$R = R_{x,\delta}\ R_{z,\alpha}\ R_{x,\theta}\ R_{z,\phi}\ R_{y,\beta}$$

$\delta = 15^o,\ \alpha = 30^o,\ \theta = 45^o,\ \phi = 60^o,\ \beta = 90^o$
$R = ?$ (Difficult ?)

Composition of Rotations:

Around fixed frame ?
Pre-multiply

Around current frame ?
Post-multiply

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Find the rotation R defined by the following basic rotations:

1. A rotation of θ about the current axis x;
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3. A rotation of α about the fixed axis z;
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5. A rotation of δ about the fixed axis x.

Solution:

$$R = R_{x,\delta} \; R_{z,\alpha} \; R_{x,\theta} \; R_{z,\phi} \; R_{y,\beta}$$

$\delta = 15^o, \;\; \alpha = 30^o, \;\; \theta = 45^o, \;\; \phi = 60^o, \;\; \beta = 90^o$

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ROBOTICS TOOLBOX!

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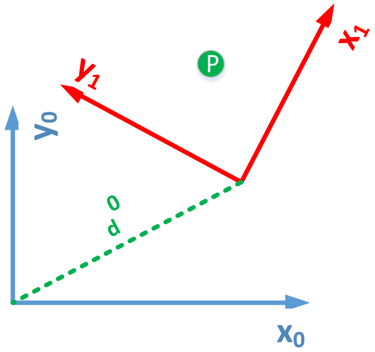
Homogeneous Transformation:

Rigid Motions:

- A rigid motion is an ordered pair (R, d) of rotation R and translation d.

$$p^0 = R_1^0 \, p^1 + d_1^0$$



Homogeneous Transformation:

Rigid Motions:

- A rigid motion is an ordered pair (R, d) of rotation R and translation d.

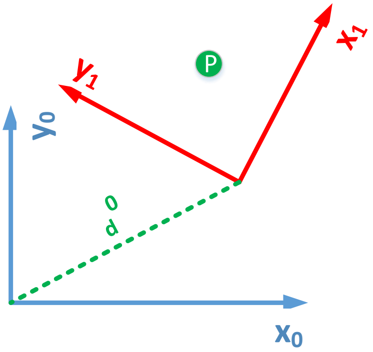
$$p^0 = R_1^0 \, p^1 + d_1^0$$

- If there are 3 frames corresponding to 2 rigid motions:

$$p^1 = R_2^1 \, p^2 + d_2^1$$
$$p^0 = R_1^0 \, p^1 + d_1^0$$

Then the overall motion is:

$$\boxed{p^0 = R_1^0 \, R_2^1 \, p^2 + R_1^0 \, d_2^1 + d_1^0}$$



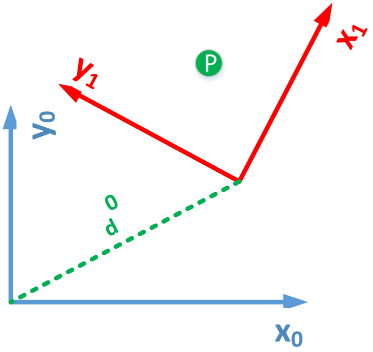
Homogeneous Transformation:

- Homogeneous Transformation is a convenient way to write the formula in a 4×4 matrix:

$$\boxed{p^0 = R_1^0\ R_2^1\ p^2 + R_1^0\ d_2^1 + d_1^0}$$

- Given a rigid motion (R, d), the 4×4-matrix T:

$$T_1^0 = \begin{bmatrix} \mathbf{R}_1^0 & \mathbf{d}_1^0 \\ 0_{1\times 3} & 1 \end{bmatrix}$$



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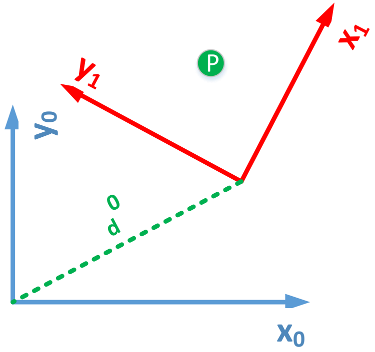
$$T_1^0 = \begin{bmatrix} \mathbf{R}_1^0 & \mathbf{d}_1^0 \\ 0_{1\times3} & 1 \end{bmatrix}$$

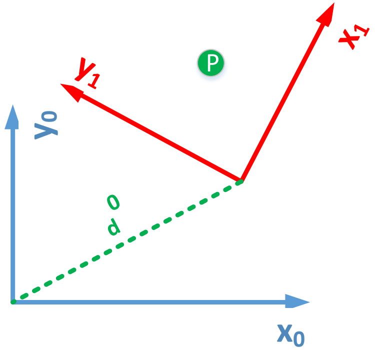
$$T_0^1 = (T_1^0)^{-1} = \begin{bmatrix} (\mathbf{R}_1^0)^T & -(\mathbf{R}_1^0)^T \mathbf{d}_1^0 \\ \mathbf{0}_{3\times1} & 1 \end{bmatrix}$$



Homogeneous Transformation:

- To use HTs in computing coordinates of point p, we need to extend the vectors of a point by one coordinate:

$$P^0 = T_1^0 \; P^1 = \begin{bmatrix} \mathbf{R}_{3*3} & \mathbf{d}_{3*1} \\ 0_{3*1} & 1 \end{bmatrix} \begin{bmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \\ \mathbf{1} \end{bmatrix}$$



For composite homogeneous transformation, the rule for **pre** and **post** multiply is valid as rotation.

Homogeneous Transformation:

Basic Homogeneous Transformation:

$$\text{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

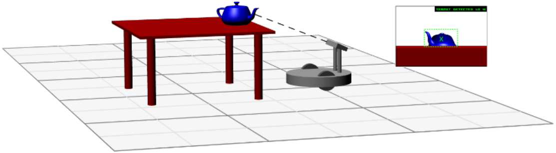
Homogeneous Transformation:

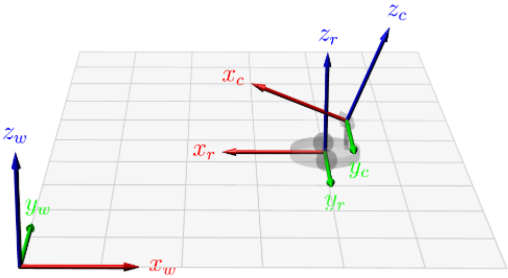
1. The teabot coordinates are expressed in camera frame, p^c.
2. To express the teabot in robot frame:

$$p^r = T^r_c \, p^c$$

3. To Express it in the world frame:

$$p^w = T^w_c \, p^c = T^w_r \, T^r_c \, p^c$$






```
T_wc = transl(4,4,.5)*trotz(180,'deg')*troty(-30,'deg')*transl(0,0,.8);
```

Homogeneous Transformation:

Example

Find homogeneous transformation matrix T that represents a rotation by angle α about the current x-axis followed by a translation of b units along the current x-axis, followed by a translation of d units along the current z-axis, followed by a rotation by angle θ about the current z-axis, is given by:

$$\text{Rot}_{x,\alpha}$$

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$$T = \text{Rot}_{x,\alpha}\ \text{Trans}_{x,b}\ \text{Trans}_{x,d}\ \text{Rot}_{z,\theta}$$

$$= \begin{bmatrix} c_\theta & -s_\theta & 0 & b \\ c_\alpha s_\theta & c_\alpha c_\theta & -s_\alpha & -ds_\alpha \\ s_\alpha s_\theta & s_\alpha c_\theta & c_\alpha & dc_\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Parameterization of Rotations:

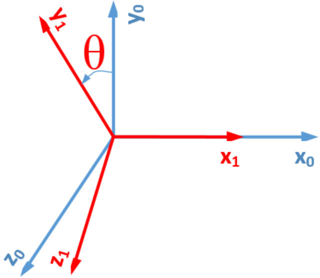
- A general rotation matrix R consists of nine elements r_{ij}.
- These nine elements are not independent quantities due to these constraints:
  1. The columns of a rotation matrix are unit vectors:

$$\sum_i r_{ij}^2 = 1, \qquad j \in \{1,2,3\}$$

  2. Columns of a rotation matrix are **mutually orthogonal**:

$$r_{1i}r_{1j} + r_{2i}r_{2j} + r_{3i}r_{3j} = 0, \qquad i \neq j$$

Example



$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

These constraints define **six independent equations** with nine unknowns, so **there are three free variables** required to define a general rotation.

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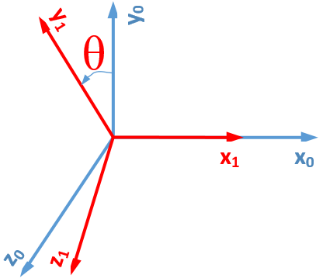

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Two ways are discussed to represent any arbitary rotations by three variable: **Euler Angles** and **Roll-Pith-Yaw** parametrization

Parameterization of Rotations: ZYZ-Euler Angles

- It is a common method of specifying a rotation matrix in terms of three independent quantities called Euler Angles $\{\phi, \theta, \psi\}$.
- Any arbitrary rotation could be represented by three successive rotations of:
  1. Rotation by ϕ about the z-axis,
  2. Followed by rotation by θ about the **current** y-axis.
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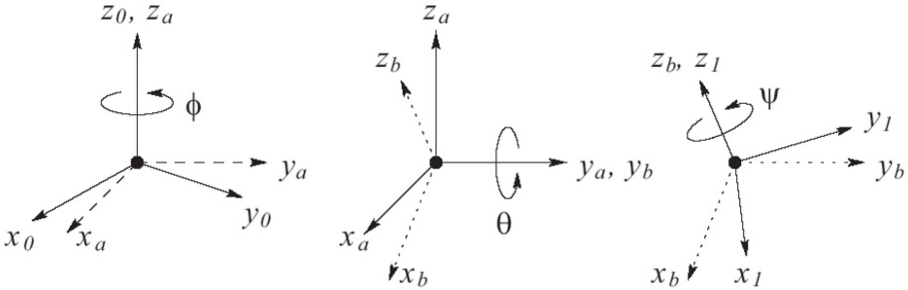

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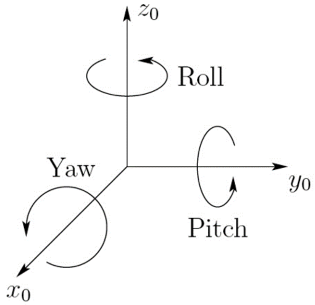
$$R_{ZYZ} = R_{z,\phi} R_{y,\theta} R_{z,\psi} = \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Parameterization of Rotations: Roll. Pitch and Yaw Angles

- A rotation matrix R could be represented as a product of three successive rotations about the **fixed coordinates**.
- These rotations define the three angles: roll, pitch, yaw, $\{\phi, \theta, \psi\}$:
  1. Rotation by ϕ about the x_0-axis,
  2. Followed by rotation by θ about the **fixed** y_0-axis.
  3. then followed by ψ about the **fixed** z_0-axis.



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  2. Followed by rotation by θ about the **fixed** y_0-axis.
  3. then followed by ψ about the **fixed** z_0-axis.

$$
\begin{aligned}
R_{XYZ} &= R_{z,\phi} R_{y,\theta} R_{x,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}
\end{aligned}
$$

Questions?

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